E9 203 Compressed Sensing & Sparse Signal Processing.

10 Mar. 2021.

Last time:

- Recovery of individual sparse vecs.
  $\Rightarrow m \geq s+1$ is sufficient
- NP hardness of ℓ_0 min.
- ℓ_1 minimization.

Today:

- ℓ_1 min. $\Rightarrow$ sparse solns.
- OMP algorithm.

Thm. 3.1 Let $A \in \mathbb{R}^{m \times N}$ be a meas. matrix with cols. $a_1, a_2, \ldots, a_N$. Assuming uniqueness of a minimizer $x^\#$ of $\boxed{(P_1): \min_{z \in \mathbb{R}^N} \|z\|_1 \text{ s.t. } Az = y}$, the system $\{a_j, j \in \text{supp}(x^\#)\}$ is LI, and $\|x^\#\|_0 \leq m$.

Proof: Suppose $\{a_j, j \in S\}$, $S \triangleq \text{supp}(x^\#)$ are LD.
Then, $\exists\, 0 \neq v \in \mathbb{R}^N$, $\text{supp}(v) \subseteq S$, s.t. $Av = 0$.
Then, for any $t \neq 0$,

$$\|x^\#\|_1 < \underbrace{\|x^\# + tv\|_1}_{\text{sub-optimal feasible pt.}} = \sum_{j \in S} |x_j^\# + t v_j|$$

$$= \sum_{j \in S} \text{sgn}(x_j^\# + t v_j)\,(x_j^\# + t v_j).$$

For t small enough, $|t| < \min_{j \in S} \frac{|x_j^\#|}{\|v\|_\infty}$

$$\text{sgn}(x_j^\# + t v_j) = \text{sgn}(x_j^\#), \quad \forall j \in S.$$

$$\|x^\#\|_1 < \sum_{j \in S} \text{sgn}(x_j^\#)(x_j^\# + t v_j)$$

$$= \sum_{j \in S} \text{sgn}(x_j^\#)\, x_j^\# + t \sum_{j \in S} \text{sgn}(x_j^\#) v_j$$

$$= \|x^\#\|_1 + t\left(\sum_{j \in S} \text{sgn}(x_j^\#)\, v_j\right)$$

contradiction $\because$ can choose $t \neq 0$ s.t. the RHS is less than $\|x^\#\|_1$. ◻

Real valued case: (P_1) is a linear program.

Slack variables $z^+, z^- \in \mathbb{R}^N$:

$$z_j^+ = \begin{cases} z_j & \text{if } z_j \geq 0 \\ 0 & \text{if } z_j < 0 \end{cases}, \quad z_j^- = \begin{cases} 0 & \text{if } z_j \geq 0 \\ -z_j & \text{if } z_j < 0. \end{cases}$$

Then (P_1) can be written as

$$\min_{z^+, z^- \in \mathbb{R}^N} \sum_{j=1}^N (z_j^+ + z_j^-) \quad \text{s.t.} \quad [A \ \ {-A}] \begin{bmatrix} z^+ \\ z^- \end{bmatrix} = y$$

$$\begin{bmatrix} z^+ \\ z^- \end{bmatrix} \geq 0.$$

Given the soln. $(z^+)^\#$, $(z^-)^\#$ of the above LP, the soln. to the pb. (P_1) is $x^\# = (z^+)^\# - (z^-)^\#$.

Complex case:

Approach 1: $\min_{z \in \mathbb{C}^N} \|z\|_0$ s.t. $Az = y$.

Write $\tilde{z} = \begin{bmatrix} z_R \\ z_I \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^{2N}$.

$u = z_R = \text{Real}(z)$, $v = z_I = \text{Imag}(z) \in \mathbb{R}^N$.

Then solve $\min_{\tilde{z} \in \mathbb{R}^{2N}} \|\tilde{z}\|_1$ s.t. $\begin{bmatrix} \text{Real}(A) & -\text{Imag}(A) \\ \text{Imag}(A) & \text{Real}(A) \end{bmatrix} \tilde{z} = \tilde{y}$,

where $\tilde{y} = \begin{bmatrix} \text{Real}(y) \\ \text{Imag}(y) \end{bmatrix} \in \mathbb{R}^{2m}$.

Approach 2: [Quadratically constrained basis pursuit]

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\|_2 \leq \eta \qquad (P_{1,\eta}).$$

Introduce $c \in \mathbb{R}^N$ s.t. $c_j \geq |z_j| = \sqrt{u_j^2 + v_j^2}$, $j \in [N]$.

Then, $(P_{1,\eta})$ is equiv. to

$$\min_{c, u, v \in \mathbb{R}^N} \sum_{j=1}^N c_j \quad \text{s.t.} \quad \left\| \begin{bmatrix} \text{Re}(A) & -\text{Im}(A) \\ \text{Im}(A) & \text{Re}(A) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} \text{Re}(y) \\ \text{Im}(y) \end{bmatrix} \right\|_2 \leq \eta$$

$$\sqrt{u_j^2 + v_j^2} \leq c_j, \quad j \in [N]$$

This is a second order cone program (SOCP).
Given its soln. $(c^\#, u^\#, v^\#)$, the soln. to $(P_{1,\eta})$ is

$$x^\# = u^\# + i\, v^\#.$$

Some related convex opt. problems:

1. Quadratically constrained basis pursuit: (QCBP)
   (BP_η): $\min_{z \in \mathbb{C}^N} \|z\|_1$ s.t. $\|Az - y\|_2 \leq \eta$.

2. Basis pursuit denoising (BPDN): For some $\lambda \geq 0$ [BPDN]
   $$\min_{z \in \mathbb{C}^N} \lambda \|z\|_1 + \|Az - y\|_2$$

3. LASSO: For some $\tau \geq 0$
   $$\min_{z \in \mathbb{C}^N} \|Az - y\|_2 \quad \text{s.t.} \quad \|z\|_1 \leq \tau$$

4. Dantzig selector $\tau \geq 0$
   $$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{s.t.} \quad \|A^H(Az - y)\|_\infty \leq \tau$$
   $$\left(\max_{j \in [N]} |a_j^H (Az - y)| : \text{small}\right)$$

Prop. (a) If x is a minimizer of BPDN w/ $\lambda > 0$, then $\exists\, \eta = \eta_x \geq 0$ s.t. x is a minimizer of QCBP.

(b) If x is a unique of QCBP, w/ $\eta \geq 0$, then $\exists\, \tau = \tau_x \geq 0$ s.t. x is a unique min. of LASSO.

(c) If x is a min. of LASSO w/ $\tau > 0$, then $\exists\, \lambda = \lambda_x \geq 0$ s.t. x is a min. of BPDN.

3.2. Greedy Methods.

Orthogonal Matching Pursuit (OMP)

Assume cols of A have unit ℓ_2 norm. $y = Ax$

Init. $S^0 = \{\emptyset\}$, $x^{(0)} = \underline{0}$, $n = 0$.

Repeat :

$$j_{n+1} = \arg\max_{j \in [N]} \left\{ \left| \left(A^H (y - Ax^{(n)}) \right)_j \right| \right\}$$

$$S^{n+1} = S^n \cup \{ j_{n+1} \}$$

$$x^{(n+1)} = \arg\min_{z \in \mathbb{C}^N} \left\{ \| y - Az \|_2 ,\ \mathrm{supp}(z) \subseteq S^{(n+1)} \right\}$$

Until a stopping criterion is met.

Output $x^{(n+1)}$.

$j_{n+1} \notin S^n$ $\because$ $A_{S^k} \in \mathbb{C}^{m \times |S^k|}$

$$\min \| A_{S^k} x_{S^k} - y \|_2^2 , \quad x_{S^k} \in \mathbb{C}^{|S^k|}$$

$$A_{S^k}^H (\underbrace{A_{S^k} x_{S^k} - y}_{-r^k,\ \text{the residual.}}) = 0$$

$$A_{S^k}^H r^k = 0 .$$

r^k orthogonal to the cols of A indexed by S^k.

$\Rightarrow$ if $j \in S^k$, it will <u>not</u> be chosen in the next iteration.